PROBABILISTIC MODEL OF CREEP AND FATIGUE INTERACTION

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1 Introduction

It has been generally recognised that the behaviour of real-life structures is determined by the aggregate effect of numerous factors which are of intrinsically stochastic nature. Therefore, dealing with issues of reliability and service life of structure the use of probabilistic (stochastic) methods can be justified.

The standard deterministic procedure of structural design generally comprises two parts. During the first phase, stresses and strains caused by external loads are calculated. These problems are addressed using methods of the theory of elasticity, theory of plasticity and other like disciplines. The second phase of the calculation involves the comparison of the calculated stresses, strains, deflections or other like parameters with values permitted under the applicable standard (or the calculation of the safety factor and its comparison with values established in the standard). The latter phase, albeit rather simple in the current computing procedures, is the most important one as it is one when the safety and cost-effectiveness of the structure are being assessed.

The above procedure may be called a conservative one as mainly the extreme values (lowest material properties, lowest wall thickness, extreme loads etc.) which are rather unlikely to occur individually and far less so in their mutual combination, are used in the calculation. The statistical concept of the safety factor has opened up new vistas for structural safety assessment including the consideration of the number of critical locations which is not possible using the conventional deterministic approach.

This is the philosophy within the framework of which the stochastic approach to material damage and damage accumulation law under creep and fatigue interaction have been conceived with special emphasis for the calculation of service life and failure risk assessment of a structure viewed as a system of critical locations.

2 Stochastic interpretation of creep material damage

Creep damage of material is usually defined as the ratio of the time of exposure and the mean or minimum time to rupture. However, through stochastic models life-time calculations can be sought more realistic. Therefore, material creep damage is defined in such a manner that the material condition is taken into account by assigning unit damage to a specific material state – material damage is unity at the moment of rupture. By means of this definition a more detailed and dynamic description of the process of damage is achieved. Hence, damage $\pi_c(t)$ at time t is defined as it follows [1]

$$\begin{aligned} \pi_c(t) &= \frac{t}{\tau(\sigma,T)} && \text{for } \tau(\sigma,T) > t \qquad (1)\\ \pi_c(t) &= 1 && \text{otherwise} \end{aligned}$$

where $\tau(\sigma,T)$ is time to rupture as random variable at constant stress σ and temperature T.

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In the following, the probabilistic interpretation of the damage accumulation law is presented. The partial damage $\Delta\pi_{ci}$ resulting from the exposure of the material at the stress σ_i and temperature T_i for time Δt_i is given by relationship (1) for $t = \Delta t_i$. The total damage at time t is given as the sum of all partial damages $\Delta\pi_{ci}$ (for exposure times Δt_i). However, considering the definition of damage according to relationship (1), damage must be redefined in the discreet manner: the condition of material before rupture (damage under unity) against the state of rupture (the damage equals unity). Then, the accumulation law shall have the form

$$\begin{aligned} \pi_c(t) &= \sum_{i=1}^{n} \frac{\Delta t_i}{\tau(\sigma_i, T_i)} && \text{if } \pi_c(t) < 1 \\ \pi_c(t) &= 1 && \text{otherwise} \end{aligned} \tag{2}$$

The random time to rupture $\tau(\sigma_i, T_i)$ can be expressed by the normalised stochastic variable Ω_{ci} [2]

$$\ln\big[\tau(\sigma_i, T_i)\big] = \delta_c\Omega_{ci} + \mu_c(\sigma_i, T_i) \tag{3}$$

where $\mu_c(\sigma_i, T_i)$ is the mean logarithmic time to rupture at stress σ_i and temperature T_i
δ_c is the standard deviation of the logarithmic time to rupture
Ω_{ci} is the random variable with unit normal probability distribution.

The random values Ω_{ci} in relationship (3) cannot be considered stochastically independent, as times to rupture $\tau(\sigma_i, T_i)$ are ones of one and the same component, as if it were exposed at parameters σ_i, T_i up to rupture. The underlying physics can be seen in the correlation of the structure and the ultimate creep material characteristic although at different conditions of temperature and stress. The following hypothesis has proved to be an adequate assumption [3]

$$\Omega_{c1} = \Omega_{c2} = \cdots = \Omega_{ci} = \cdots = \Omega_{cn} = \Omega_c \tag{4}$$

According to relationship (2) and in pursuance of relationships (3) and (4), the law of accumulation of material damage has the form

$$\begin{aligned} \pi_c(t) &= \sum_{i=1}^{n} \frac{\Delta t_i}{\exp(\delta_c\Omega_c + \mu_{ci})} = \exp(-\delta_c\Omega_c) \sum_{i=1}^{n} \frac{\Delta t_i}{t_c(\sigma_i, T_i)} && \text{for } \pi_c(t) < 1 \\ \pi_c(t) &= 1 && \text{otherwise} \\ t &= \sum_{i=1}^{n} \Delta t_i \end{aligned} \tag{5}$$

where $t_c(\sigma_i, T_i) = \exp(\mu_{ci})$ is the delogarithmised mean logarithmic time to rupture
$\mu_{ci} = \mu_c(\sigma_i, T_i)$ cf. relationship (3).

3 Fatigue material damage as a random variable

When the time or number of cycles to crack initiation are calculated the terms of material damage and damage accumulation law are also used. To use a similar concept in stochastic calculations, these notions have to be redefined to account of the random nature of the fatigue characteristics. Likewise, the subsequent growth of the fatigue crack must be understood to be an intrinsically stochastic process with which crack initiation occurs once the critical state is reached.

Material damage in fatigue process is usually defined as the ratio of the number of cycles N and the mean number of cycles to crack initiation or a minimum value. However, the damage so defined disregards the possible variability of the material properties and for a given number of cycles one and the same value is always obtained regardless of the actual strength of the material. To eliminate this discrepancy, fatigue damage $\pi_f(N)$ is defined in a manner similar to that use in the case of creep

$$\begin{aligned} &\pi_f(N) = \frac{N}{\nu(\Delta\sigma, T)} && \text{for } \nu(\Delta\sigma, T) > N \\ &\pi_f(N) = 1 && \text{otherwise} \end{aligned} \tag{6}$$

where N is the number of cycles
$\nu(\Delta\sigma, T)$ is the number of cycles to crack initiation as stochastic variable
$\Delta\sigma$ is the stress amplitude and T is the temperature.

The partial damage $\Delta\pi_{fi}$ arising from the exposure of the material at temperature T_i and stress amplitude $\Delta\sigma_i$ for the number of cycles ΔN_i follows from equation (6) for $N = \Delta N_i$. While the total damage after N cycles will be the sum total of the partial damages, given by the definition of damage according to rel. (6), two situations must be differentiated between. The condition before crack initiation (damage under unity) and the condition after the crack has initiated (damage equals unity). Thereafter the damage accumulation law shall have the form

$$\begin{aligned} &\pi_f(N) = \sum_{i=1}^{n} \frac{\Delta N_i}{\nu(\Delta\sigma_i, T_i)} && \text{for } \pi_f(N) < 1 \\ &\pi_f(N) = 1 && \text{otherwise} \end{aligned} \tag{7}$$

The random number of cycles to crack initiation $\nu(\Delta\sigma_i, T_i)$ can be expressed again using by the normalised random variable Ω_{fi} [4]

$$\ln\left[\nu(\Delta\sigma_i, T_i)\right] = \delta_f \Omega_{fi} + \mu_f(\Delta\sigma_i, T_i) \tag{8}$$

where $\mu_f(\Delta\sigma_i, T_i)$ is the mean of the logarithm of the cycles number to crack initiation at stress amplitude $\Delta\sigma_i$ and temperature T_i
δ_f is the standard deviation of the logarithm of the cycles number to crack initiation
Ω_{fi} is a random variable with unit normal probabilistic distribution.

However, the random variables Ω_{fi} in relationship (8) are not stochastically independent. As if they were, the result of the calculation would depend on the number of division n, the variables, being the cycles to crack initiation of one and the same component as if it were exposed to parameters $\Delta\sigma_i, T_i$ to crack initiation. The following assumption has proved adequate [4,5]

$$\Omega_{f1} = \Omega_{f2} = \cdots = \Omega_{fi} = \cdots = \Omega_{fn} = \Omega_f \tag{9}$$

Using equations (8) and (9), relationship (7) has the form

$$\begin{aligned} &\pi_f(N) = \sum_{i=1}^{n} \frac{\Delta N_i}{\exp(\delta_f \Omega_f + \mu_{fi})} = \exp(-\delta_f \Omega_f) \sum_{i=1}^{n} \frac{\Delta N_i}{N_f(\Delta\sigma_i, T_i)} && \text{if } \pi_f(N) < 1 \\ &\pi_f(N) = 1 && \text{otherwise} \\ &\quad N = \sum_{i=1}^{n} \Delta N_i \end{aligned} \tag{10}$$

where $N_f(\Delta\sigma_i, T_i) = \exp(\mu_{fi})$ is the delogaritmised mean of the logarithm of the cycles number to crack initiation, $\mu_{fi} = \mu_f(\Delta\sigma_i, T)$ – cf. relationship (8).

4 Material damage accumulation under fatigue and creep interaction

In above Paragraphs 2 and 3 the stochastic interpretation of material damage accumulation under fatigue and creep was presented. As in many cases these two processes run parallel, the stochastic form of the law of damage accumulation under interaction of these two processes including the issue of synergism is derived.

According to Paragraph 2, the damage caused by creep can be described as a random process by the relationship (5) and the fatigue damage (according to Paragraph 3) is expressed by the relationship (10).

Using relationships (5) and (10), the material damage due to creep and fatigue can be modified into another form which it can be seen that the damage is a function of the random variables Ω_c and Ω_f and time t

$$\pi(t) = \pi_c(t) + \pi_f\big[N(t)\big] = C(t, \Omega_c) + F\big[N(t), \Omega_f\big] \tag{11}$$

where

$$\begin{aligned} C(t, \Omega_c) &= \exp(-\delta_c \Omega_c) \sum_{i=1}^{n} \frac{\Delta t_i}{t_c(\sigma_i, T_i)} \\ F(t, \Omega_f) &= \exp(-\delta_f \Omega_f) \sum_{i=1}^{n} \frac{N(\Delta t_i)}{N_f(\Delta\sigma_i, T_i)} \\ N(t) = N &= \sum_{i=1}^{n} N(\Delta t_i), \qquad N(\Delta t_i) \text{ is cycles number in time interval } \Delta t_i \end{aligned} \tag{12}$$

Under creep and fatigue interaction, the condition of ultimate material damage (rupture or crack initiation) can be considered as assumed in above Paragraphs 2 and 3 i.e. when the total damage $\pi(t)$ according to relationship (11) reaches the value of one. However, there have been many references to other values [6, 7]. This matter is not dealt with here in terms of the material. However, the critical state can be described by the relationship [8]

$$\pi_c(t) + \pi_f\big[N(t)\big] = D \tag{13}$$

which can be expressed as a limit curve D_1 defining the safe-fail and failure areas: i.e. there exists an equation bounding the corresponding areas in a diagram with the fatigue damage π_f and the creep damage π_c used as co-ordinates (cf. fig. 1).

In view of the above concept of the limit curve D_1, the damage according to relationship (11) need not necessarily equal unity at the limit curve. Therefore the damage at creep and fatigue interaction will be defined as the ratio of the distance of the point given by the fatigue and creep damage $[\pi_f(t), \pi_c[N(t)]]$ and the corresponding point $[p_{f1}, p_{c1}]$ at the limit curve D_1 from the origin – cf. fig. 2.

The above definition can be interpreted as the normalisation of the limit states given by the curve D_1 so that the ultimate material damage condition as a random variable is always expressed by unity as assumed in above Paragraphs 2 and 3. Thereafter, the total damage is defined by the relationship [9]

$$\pi(t) = \sqrt{\frac{\pi_f^2(t) + \pi_c^2[N(t)]}{p_{f1}^2 + p_{c1}^2}} \text{for } \pi(t) < 1,\ \pi(t) = 1 \text{ otherwise} \tag{14}$$

The co-ordinates of the point $[p_{f1} + p_{c1}]$ can be obtained by solving the equations

$$\frac{\pi_c(t)}{\pi_f[N(t)]} p_{f1} = D_1(p_{f1}) \quad \text{and} \quad p_{c1} = p_{f1} \frac{\pi_c(t)}{\pi_f[N(t)]} \tag{15}$$

The definition of damage so introduced can be visualised by lines of constant damage as shown in fig. 3.

The total random damage $\pi(t)$ is determined by the random variables Ω_c and Ω_f. Naturally, the issue of stochastic dependence of these two variables arises as in terms of material structure. The two damage processes can act effectively one against the other. Coarse grain size, while potentially contributing to higher heat resistance, can result in compromised fatigue characteristics. The issue in question is essentially one of synergism. In terms of probabilistic interpretation this situation can be viewed as stochastic dependence or independence of the two random variables.

For the two random variables Ω_f and Ω_c the Gaussian probability distribution has been assumed and hence their simultaneous distribution can be assumed to be the two-dimensional normal distribution of density $h(\Omega_c, \Omega_f)$ [10]

$$h(\Omega_c, \Omega_f) = \frac{1}{2\pi\sqrt{1-r^2}} \exp\left[-\frac{1}{2(1-r^2)}\left[\Omega_c^2 + 2r\Omega_c\Omega_f + \Omega_f^2\right]\right] \tag{16}$$

where r is the coefficient of correlation of the random quantities Ω_c, Ω_f.

Thereby, the rate of material damage in the above probabilistic model of interaction of creep and fatigue material damage can be controlled: positive and negative coefficients of correlation accelerating and decelerating the damage rate respectively.

5 Application of the random model

For probabilistic calculation it is necessary further to derive probabilistic distribution function the risk of failure and material damage. As the trajectories of the accumulated damage are non-decreasing, the point $[\pi_f[N(t)], \pi_c(t)]$ defining the damage occurs in the safe-fail condition if it is located below the limit curve D_1 i.e. if it holds

$$\pi_c(t) < D_1\big[\pi_f\big[N(t)\big]\big] \tag{17}$$

The risk of failure $R(t)$ before time t with interaction of creep and fatigue is given by the probability of the complementary event according to relationship (17)

$$R(t) = 1 - P\left\{\pi_c(t) < D_1\big[\pi_f\big[N(t)\big]\big]\right\} \tag{18}$$

The distribution function of damage is established from relationship

$$P\big[\pi_c(t) < p\big] = P\left\{\pi_c(t) < D_p\big[\pi_f\big[N(t)\big]\big]\right\} \tag{19}$$

where D_p is the curve corresponding to damage p, $0 \leq p \leq 1$.

According to the definition of damage by relationship (14) and by the use of limit damage curve D_1, the curves D_p corresponding to damage p follow from the solution of the below sub-system of equations

$$p^2\left[p_{fD}^2 + D_1^2(p_{fD})\right] = \pi_f^2\left[1 + \frac{D_1^2(p_{fD})}{p_{fD}^2}\right] \tag{20}$$

where from the co-ordinate p_{fD} results as a function of the input value of damage p and π_f. The value of the damage curve D_p corresponding to the level p is obtained from equation

$$\pi_c = D_p(\pi_f) = \pi_f\frac{D_1(p_{fD})}{p_{fD}} \tag{21}$$

6 Conclusion

The definition of stochastic material damage in creep and fatigue was introduced, the time to rupture and number of cycles to crack initiation being viewed as random variables. In pursuance of this definition the damage accumulation laws were derived.

The probabilistic interpretation of damage accumulation under creep and fatigue interaction is of major significance. A new concept of damage was introduced whereby the ultimate condition of material damage given by the limit curve is defined by unity, analogously to the separate damage definitions for material damage by creep and fatigue.

The probability distribution function of material damage and risk of failure under creep and fatigue interaction were derived for operational reliability calculation [11].

The interactive probabilistic model is determined by the two random variables expressing the stochastic nature of the creep and fatigue processes. By their correlation, the problem of synergism can be dealt with in probabilistic terms through the two-dimensional probability distribution of the variables mentioned above.

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References

[1] Bína V., Hakl J., Vlasák T., Bielak O., Korouš J.: Application of Data on Creep Behaviour in Life-Time Assessment. Proceedings of the Fifth Int. Charles Parsons Turbine Conference, July 2000, Churchil College, Cambridge 2000, p.1023-1035.

[2] Zchokke H.: Archiv fűr das Eisenhuttenwesen 28, 1957, p. 726-732.

[3] Pech R.: Vliv proměnného napětí a teploty při tečení (The influence of varying stress and temperature under creep – in Czech). Report SVUM Z-70-2479, Praha 1970.

[4] Provan J. W. : Probabilistic Fracture Mechanics and Reliability. Martinus Nijhoff Publisher, Dordrecht 1987.

[5] Němec J., Sedláček J. : Statistické základy pevnosti konstrukcí 1 (Statistical theory of structure strength – in Czech). Academia, Praha 1982.

[6] Manson S. S., Halford G. R. , Hirschberg M. H. : Creep-Fatigue Analisis by Strain-Range Partitioning. Design for Elevated Temperature Enviroment, ASME, New York 1971, p. 12-24.

[7] Vodsedálek J., Vystyd M., Pech R.: Vlastnosti a použití žáropevnch ocelí a slitin. (Properties and the use of creep resistant steels and alloys – in Czech), SNTL, Praha 1974.

[8] Thacker B. H., Popelar C. F., McClung R. C.: A New Probabilistic Method for Predicting the Long-Life Reliability of Pipelines. The Int. Conf. On Pipeline Reliability. Calgary, Metals Technology Laboratories, CANMET, Ottawa, Gulf Publishing Company, London 1992.

[9] Bielak O., Bína V., Korouš J.: Pravděpodobnostní přístupy hodnocení provozní spolehlivosti systémů, jejichž prvky jsou poškozovány creepem a únavou. (Probabilistic approach to assessment of operational reliability of structure parts damaged by creep – in Czech). Report BiSAFE Z-96-067, Grant project GAČR No. 106/94/0429. Praha 1996.

[10] Rao C. R.: Linear Statistical Method and its Application. John Willey and Sons, INC. New York – Sydney 1968.

[11] Korouš J., Bína V., Bielak O.: Stress Redistribution and Life-Time of Y-Piece by Stochastic Creep Approach. 14th Symposium Danubia-Adria on Experimental Methods in Solid Mechanics, October 1997, Poreč-Croatia, p. 163-164.

List of symbols

T	temperature [K]
$\sigma, \Delta\sigma$	stress and stress amplitude respectively [MPa]
$t, \Delta t$	time and time increment respectively [h]
p	damage [–]
$\pi_c(t)$	creep damage as stochastic process at time t [–]
$t_c(\sigma, T)$	mean value of the time to rupture at constant stress s and temperature T [h]
t_{ci}	mean value of the time to rupture at constant stress si and temperature T_i [h]
$\tau(\sigma, T)$	time to rupture as random variable at constant stress s and temperature T [h]
$\mu_c(\sigma, T)$	mean value of the logarithmic time to rupture at constant stress s and temperature T [–]
$N, N(t)$	number of cycles, number of cycles at time t respectively [–]
ΔN	cycle increment [-]
$\pi_f[N(t)]$	fatigue damage as stochastic process at time t [–]
$N_f(\Delta\sigma, T)$	mean value of the cycles number to crack initiation at constant stress amplitude $\Delta\sigma$ and temperature T [-]
N_{fi}	mean value of the cycles number to crack initiation at constant stress amplitude $\Delta\sigma_i$ and temperature T_i [–]
$\nu(\Delta\sigma, T)$	number of cycles to crack initiation as stochastic variable at constant stress amplitude $\Delta\sigma$ and temperature T [–]
$\mu_f(\Delta\sigma, T)$	mean value of the logarithm of the cycles number to crack initiation at constant stress amplitude $\Delta\sigma$ and temperature T [–]
δ_f	standard deviation of the logarithm of the cycles number to crack initiation [–]
δ_c	standard deviation of the logarithmic time to rupture [–]
$P(*)$	probability of the statement indicated by (*)
Ω_f, Ω_c	random variables having unit normal probabilistic distribution [–]
$h(\Omega_f, \Omega_c)$	two-dimensional Gaussian probability density function
r	correlation coefficient of the random variables Ω_f, Ω_c [–]
D_1	limit failure curve under creep and fatigue interaction
D_p	constant damage p curve under creep and fatigue interaction

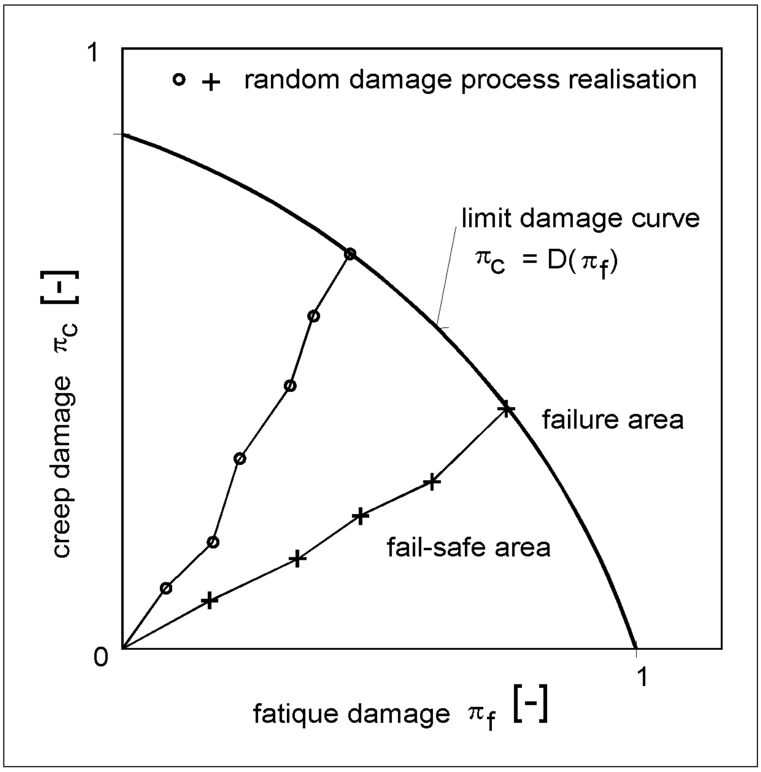

Figure 1: Stochastic interpretation of material damage under creep and fatigue interaction

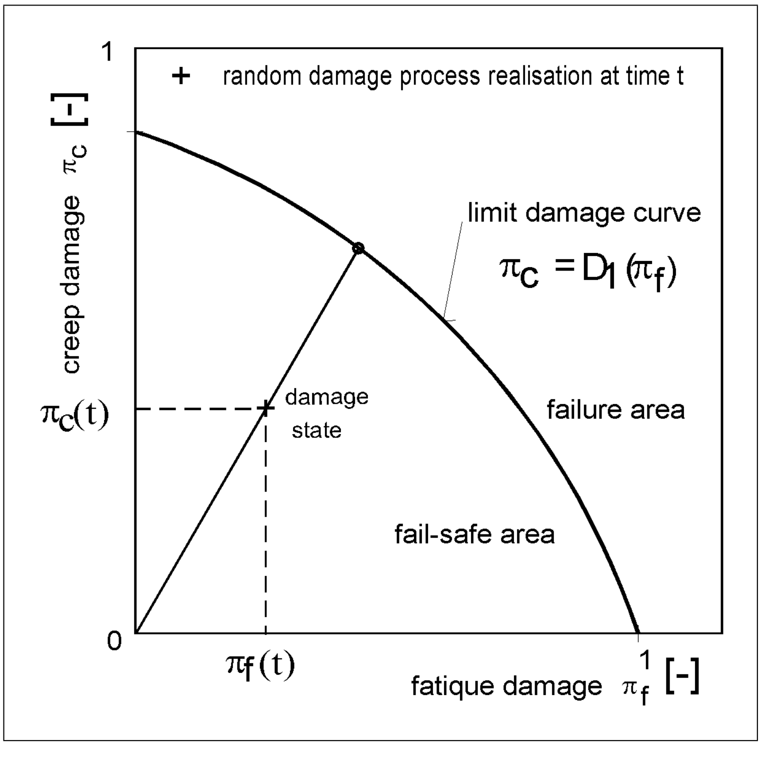

Figure 2: Definition of material damage under creep and fatigue interaction

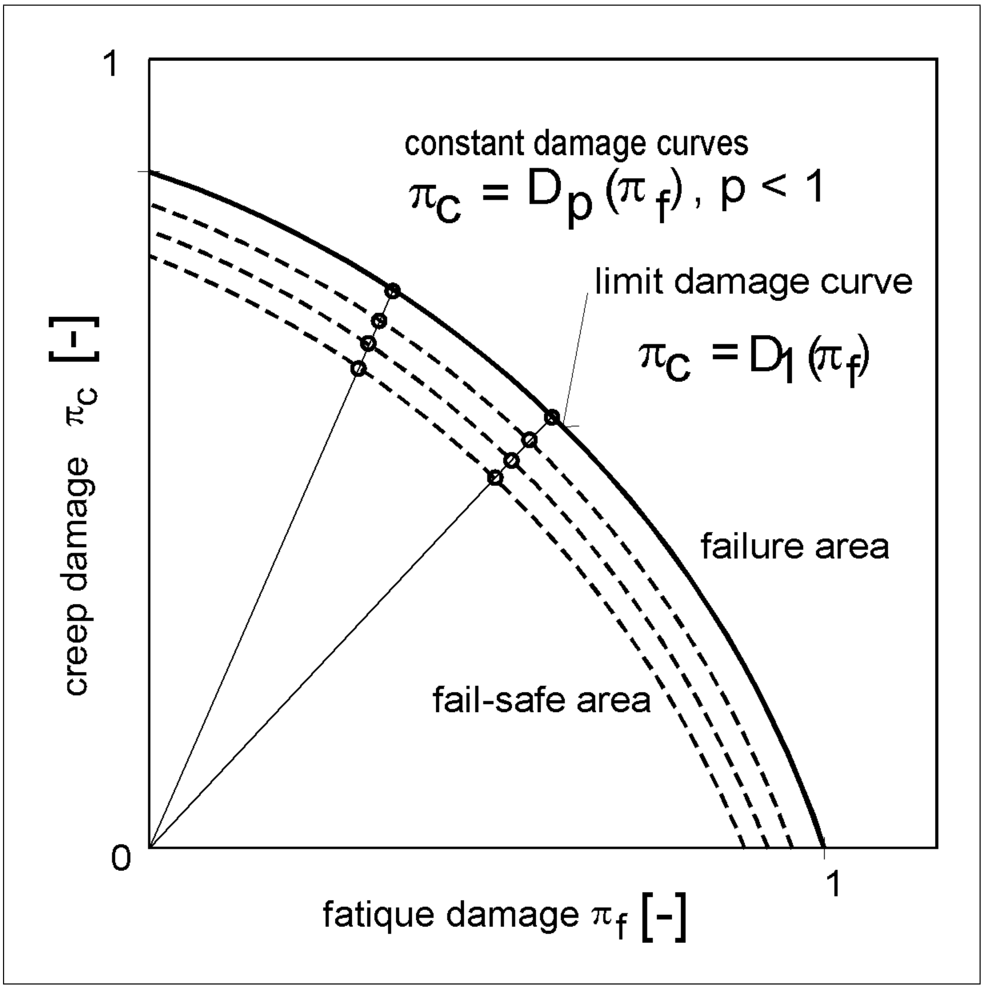

Figure 3: Constant damage curves under creep and fatigue interaction